Inner fluctuations of the spectral action

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We dedicate this paper to Daniel Kastler on his eightieth birthday

Abstract

We prove in the general framework of noncommutative geometry that the inner fluctuations of the spectral action can be computed as residues and give exactly the counterterms for the Feynman graphs with fermionic internal lines. We show that for geometries of dimension less than or equal to four the obtained terms add up to a sum of a Yang–Mills action with a Chern–Simons action.

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1. Introduction

The spectral action is defined as a functional on noncommutative geometries. Such a geometry is specified by a fairly simple data of operator theoretic nature, namely a spectral triple

$$(\mathcal{A}, \mathcal{H}, D), \tag{1.1}$$

where $\mathcal{A}$ is a noncommutative algebra with involution $*$, acting in the Hilbert space $\mathcal{H}$ while D is a self-adjoint operator with compact resolvent and such that

$$[D, a] \quad \text{is bounded } \forall a \in \mathcal{A}. \tag{1.2}$$

Additional structures such as the $\mathbb{Z}/2\mathbb{Z}$ grading γ in the even case and the real structure J of $\mathcal{H}$ will play little role below, but can easily be taken into account.

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The spectral action fulfills two basic properties

- It only depends upon the spectrum of D.
- It is additive for direct sums of noncommutative geometries.

It is given in general by the expression

$$\text{Trace}(f(D/\Lambda)), \tag{1.3}$$

where f is a positive even function of the real variable and the parameter Λ fixes the mass scale. The dimension of a noncommutative geometry is not a number but a spectrum, the *dimension spectrum* (*cf.* [6]) which is the subset Π of the complex plane $\mathbb{C}$ at which the spectral functions have singularities. Under the hypothesis that the dimension spectrum is *simple i.e.* that the spectral functions have at most simple poles, the residue at the pole defines a far reaching extension (*cf.* [6]) of the fundamental integral in noncommutative geometry given by the Dixmier trace (*cf.* [3]). This extends to the framework of spectral triples the Wodzicki residue (originally defined for pseudodifferential operators on standard manifolds) as a trace on the algebra of operators generated by $\mathcal{A}$ and powers of D so that

$$P \to \fint P \in \mathbb{C}, \quad \fint P_1 P_2 = \fint P_2 P_1. \tag{1.4}$$

Both this algebra and the functional (1.4) do not depend on the detailed knowledge of the *metric* defined by D and the residue is unaltered by a change $D \to D'$ of D such that the difference

$$\text{Log } D' - \text{Log } D,$$

is a *bounded* operator with suitable regularity. In other words the residue only depends on the quasi-isometry class of the noncommutative metric.

In this generality the spectral action (1.3) can be expanded in decreasing powers of the scale Λ in the form

$$\text{Trace}(f(D/\Lambda)) \sim \sum_{k \in \Pi^+} f_k \Lambda^k \fint |D|^{-k} + f(0)\zeta_D(0) + o(1), \tag{1.5}$$

where Π^+ is the positive part of the dimension spectrum Π. The function f only appears through the scalars

$$f_k = \int_0^\infty f(v) v^{k-1} \mathrm{d}v. \tag{1.6}$$

One lets

$$\zeta_D(s) = \text{Tr}(|D|^{-s}), \tag{1.7}$$

and regularity at $s = 0$ is assumed.

Both the gauge bosons and the Feynman graphs with fermionic internal lines can be readily defined in the above generality of a noncommutative geometry $(\mathcal{A}, \mathcal{H}, D)$ (*cf.* [2]). Indeed, as briefly recalled at the beginning of Section 2, the inner fluctuations of the metric coming from the Morita equivalence $\mathcal{A} \sim \mathcal{A}$ generate perturbations of D of the form $D \to D' = D + A$ where the A plays the role of the gauge potentials and is a self-adjoint element of the bimodule

$$\Omega^1_D = \left\{ \sum a_j [D, b_j]; a_j, b_j \in \mathcal{A} \right\}. \tag{1.8}$$

The line element $\mathrm{d}s = D^{-1}$ plays the role of the fermion propagator so that the value $U(\Gamma_n)$ of one loop graphs Γ_n with fermionic internal lines and n external bosonic lines (such as the triangle graph of Fig. 1) is easy to obtain and given at the formal level by,

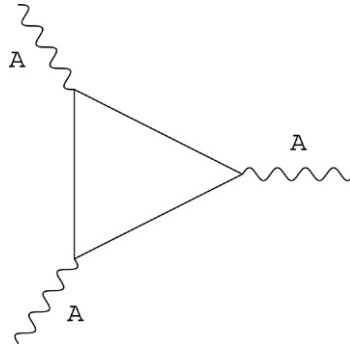$$U(\Gamma_n) = \text{Tr}((AD^{-1})^n).$$
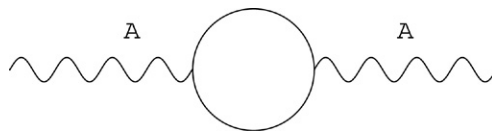
Fig. 1. The triangle graph.
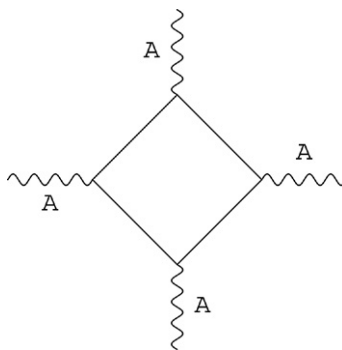

Fig. 2. The self-energy graph.

Fig. 3. The quartic graph.

These graphs diverge in dimension 4 for $n \leq 4$ and the residue at the pole in dimensional regularization can be computed and expressed as

$$\fint (AD^{-1})^n,$$

as will be shown in [5].

In this paper we analyze how the spectral action behaves under the inner fluctuations. The main results are

- In dimension 4 the variation of the spectral action under inner fluctuations gives the local counterterms for the fermionic graphs of Figs. 4, 2, 1 and 3 respectively

$$\zeta_{D+A}(0) - \zeta_D(0) = - \fint AD^{-1} + \frac{1}{2} \fint (AD^{-1})^2 - \frac{1}{3} \fint (AD^{-1})^3 + \frac{1}{4} \fint (AD^{-1})^4,$$

- Assuming that the tadpole graph of Fig. 4 vanishes the above variation is the sum of a Yang–Mills action and a Chern–Simons action relative to a cyclic 3-cocycle on $\mathcal{A}$.
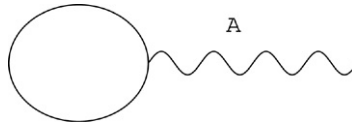
Fig. 4. The tadpole graph.

As a corollary, combining both results we obtain that the variation under inner fluctuations of the scale independent terms of the spectral action is given (*cf.* Theorem 3.5 for precise notations) in dimension 4 by

$$\zeta_{D+A}(0) - \zeta_D(0) = \frac{1}{4} \int_{\tau_0} (\mathrm{d}A + A^2)^2 - \frac{1}{2} \int_\psi \left(A\mathrm{d}A + \frac{2}{3} A^3 \right). \tag{1.9}$$

The conceptual meaning of the above tadpole condition is that the original noncommutative geometry $(\mathcal{A}, \mathcal{H}, D)$ is a critical point for the (Λ-independent part of the) spectral action, which is a natural hypothesis. The functional τ_0 is a Hochschild 4-cocycle but in general not a cyclic cocycle. In particular, as explained in detail in [3] Chapter VI, the expression

$$\int_{\tau_0} (\mathrm{d}A + A^2)^2, \tag{1.10}$$

coincides with the Yang–Mills action functional provided that $\tau_0 \geq 0$ *i.e.* that τ_0 is a positive Hochschild cocycle. The Hochschild cocycle τ_0 cannot be cyclic unless the expression (1.10) vanishes.

We show at the end of the paper that the cyclic cohomology class of the cyclic three cocycle ψ is determined modulo the image of the boundary operator B and that the pairing of ψ with the K_1-group is trivial. This shows that under rather general assumptions one can eliminate ψ by a suitable redefinition of τ_0 (see Proposition 3.7).

The meaning of the vanishing of ψ together with positivity of τ_0 is that the original noncommutative geometry $(\mathcal{A}, \mathcal{H}, D)$ is at a *stable* critical point as far as the inner fluctuations are concerned. In fact it gives in that case an absolute minimum for the (scale independent terms of the) spectral action in the corresponding class modulo inner fluctuations. We end the paper with the corresponding open questions : elimination of ψ and positivity of the 4-cocycle τ_0.

2. Inner fluctuations of the metric and the spectral action

The inner fluctuations of the noncommutative metric appear through the simple issue of Morita equivalence. Indeed let $\mathcal{B}$ be the algebra of endomorphisms of a finite projective (right) module $\mathcal{E}$ over $\mathcal{A}$

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}). \tag{2.1}$$

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ one easily gets a representation of $\mathcal{B}$ in the Hilbert space

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}.$$

But to define the analogue D' of the operator D for $\mathcal{B}$ requires the choice of a *Hermitian connection* on $\mathcal{E}$. Such a connection ∇ is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ satisfying the following rules [3]

$$\nabla(\xi a) = (\nabla \xi)a + \xi \otimes \mathrm{d}a, \quad \forall \xi \in \mathcal{E}, a \in \mathcal{A}, \tag{2.2}$$

$$(\xi, \nabla \eta) - (\nabla \xi, \eta) = \mathrm{d}(\xi, \eta), \quad \forall \xi, \eta \in \mathcal{E}, \tag{2.3}$$

where $\mathrm{d}a = [D, a]$ and where $\Omega^1_D \subset \mathcal{L}(\mathcal{H})$ is the $\mathcal{A}$-bimodule (1.8). The operator D' is then given by

$$D'(\xi \otimes \eta) = \xi \otimes D\eta + \nabla(\xi)\eta. \tag{2.4}$$

Any algebra $\mathcal{A}$ is Morita equivalent to itself and when one applies the above construction with $\mathcal{E} = \mathcal{A}$ one gets the inner deformations of the spectral geometry. These replace the operator D by

$$D \to D + A, \tag{2.5}$$

where $A = A^*$ is an arbitrary selfadjoint element of Ω_D^1 where we disregard the real structure for simplicity. To incorporate the real structure one replaces the algebra $\mathcal{A}$ by its tensor product $\mathcal{A} \otimes \mathcal{A}^o$ with the opposite algebra.

2.1. Pseudodifferential calculus

As developed in [6] one has under suitable regularity hypothesis on the spectral geometry $(\mathcal{A}, \mathcal{H}, D)$ an analogue of the pseudodifferential calculus. We briefly recall the main ingredients here. We say that an operator T in $\mathcal{H}$ is *smooth* iff

$$t \to F_t(T) = e^{it|D|} T e^{-it|D|} \in C^\infty(\mathbb{R}, \mathcal{L}(\mathcal{H})), \tag{2.6}$$

and let OP^0 be the algebra of smooth operators. Any smooth operator T belongs to the domains of δ^n, where the derivation δ is defined by

$$\delta(T) = |D|T - T|D| = [|D|, T]. \tag{2.7}$$

The analogue of the Sobolev spaces are given by

$$\mathcal{H}_s = \text{Dom}|D|^s \quad s \geq 0, \quad \mathcal{H}_{-s} = (\mathcal{H}_s)^*, \quad s < 0.$$

For any smooth operator T one has (*cf.* [6]) $T\mathcal{H}_s \subset \mathcal{H}_s$ and we let

$$OP^\alpha = \{T; |D|^{-\alpha} T \in OP^0\}.$$

We work in dimension ≤ 4 which means that $ds = D^{-1}$ is an infinitesimal of order $\frac{1}{4}$ and thus that for $N > 4$, OP^{-N} is inside trace class operators. In general we work modulo operators of large negative order, *i.e.* mod OP^{-N} for large N. We let $\mathcal{D}(\mathcal{A})$ be the algebra generated by $\mathcal{A}$ and D considered first at the formal level. The main point is the following lemma [6] which allows us to multiply together *pseudodifferential* operators of the form

$$PD^{-2n}, \quad P \in \mathcal{D}(\mathcal{A}). \tag{2.8}$$

One lets $\nabla(T) = D^2 T - T D^2$.

Lemma 2.1 (*[6]*). *Let* $T \in OP^0$.

(a) $\nabla^n(T) \in OP^n \, \forall n \in \mathbb{N}$.
(b) $D^{-2}T = \sum_0^n (-1)^k \nabla^k(T) D^{-2k-2} + (-1)^{n+1} D^{-2} \nabla^{n+1}(T) D^{-2n-2}$.
(c) *The remainder* $R_n = D^{-2} \nabla^{n+1}(T) D^{-2n-2}$ *belongs to* $OP^{-(n+3)}$.

Proof. (a) The equality

$$|D|T|D|^{-1} = T + \beta(T), \quad \beta(T) = \delta(T)|D|^{-1}, \tag{2.9}$$

shows that for $T \in OP^0$ one has

$$D^2 T D^{-2} = T + 2\beta(T) + \beta^2(T) \in OP^0. \tag{2.10}$$

Similarly one has,

$$D^{-2} T D^2 \in OP^0. \tag{2.11}$$

This shows that in the definition of OP^α one can put $|D|^{-\alpha}$ on either side.

To prove (a) we just need to check that $\nabla(T) \in OP^1$ and then proceed by induction. We have $\nabla(T) = D^2 T - T D^2 = (D^2 T D^{-2} - T) D^2 = (2\beta(T) + \beta^2(T)) D^2 = 2\delta(T)|D| + \delta^2(T)$,

$$\nabla(T) = 2\delta(T)|D| + \delta^2(T), \tag{2.12}$$

which belongs to OP^1.

(b) For $n = 0$ the statement follows from

$$D^{-2}T = T D^{-2} - D^{-2} \nabla(T) D^{-2}. \tag{2.13}$$

Next assume we proved the result for $(n - 1)$. To get it for n we must show that

$$(-1)^n \nabla^n(T) D^{-2n-2} + (-1)^{n+1} D^{-2} \nabla^{n+1}(T) D^{-2n-2} = (-1)^n D^{-2} \nabla^n(T) D^{-2n}. \tag{2.14}$$

Multiplying by D^{2n} on the right, with $T' = (-1)^n \nabla^n(T)$, we need to show that

$$T' D^{-2} - D^{-2} \nabla(T') D^{-2} = D^{-2} T',$$

which is (2.13).

(c) Follows from (a).   □

Thus when working mod OP^{-N} for large N one can write

$$D^{-2} T \sim \sum_0^\infty (-1)^k \nabla^k(T) D^{-2k-2}, \tag{2.15}$$

and this allows us to compute the product in the algebra $\Psi\mathcal{D}$ of operators which, modulo OP^{-N} for any N, are of the form (2.8). Such operators will be called pseudodifferential.

2.2. The operator $\mathrm{Log}\,(D + A)^2 - \mathrm{Log}\,(D^2)$

We let A be a gauge potential,

$$A = \sum a_i [D, b_i]; a_i, b_i \in \mathcal{A}, \quad A = A^*, \tag{2.16}$$

and we consider the operator X defined from the square of the self-adjoint operator $D + A$,

$$(D + A)^2 = D^2 + X, \quad X = AD + DA + A^2. \tag{2.17}$$

The following lemma is an adaptation to our set-up of a classical result in the pseudodifferential calculus on manifolds,

Lemma 2.2.

$$Y = \mathrm{Log}\,(D + A)^2 - \mathrm{Log}\,(D^2) \in \Psi\mathcal{D} \cap OP^{-1}.$$

Proof. We start with the equality $(a > 0)$

$$\mathrm{Log}\,a = \int_0^\infty \left(\frac{1}{\lambda + 1} - \frac{1}{\lambda + a} \right) \mathrm{d}\lambda, \tag{2.18}$$

and apply it to both D^2 and $(D + A)^2 = D^2 + X$ to get,

$$Y = \int_0^\infty \left(\frac{1}{\lambda + D^2} - \frac{1}{\lambda + D^2 + X} \right) \mathrm{d}\lambda. \tag{2.19}$$

One has

$$(\lambda + D^2 + X)^{-1} = ((1 + X(D^2 + \lambda)^{-1})(D^2 + \lambda))^{-1} = (D^2 + \lambda)^{-1}(1 + X(D^2 + \lambda)^{-1})^{-1},$$

and one can expand,

$$(1 + X(D^2 + \lambda)^{-1})^{-1} = \sum_0^\infty (-1)^n (X(D^2 + \lambda)^{-1})^n. \tag{2.20}$$

In this expansion the remainder is, up to sign,

$$(X(D^2 + \lambda)^{-1})^{n+1}(1 + X(D^2 + \lambda)^{-1})^{-1} = R_n(\lambda). \tag{2.21}$$

Here $X \in OP^1$ by construction so that a rough estimate of the order of the remainder is given by

$$\int X^{n+1}(D^2 + \lambda)^{-(n+1)} \mathrm{d}\lambda \sim X^{n+1}(D^2)^{-n} \sim |D|^{n+1-2n}. \tag{2.22}$$

Now in Lemma 2.1(b) we can use $D^2 + \lambda$ instead of D^2. This does not alter ∇ since

$$[D^2 + \lambda, T] = [D^2, T], \tag{2.23}$$

and we thus get,

$$(D^2 + \lambda)^{-1} T = \sum_0^n (-1)^k \nabla^k(T)(D^2 + \lambda)^{-(k+1)} + (-1)^{n+1}(D^2 + \lambda)^{-1} \nabla^{n+1}(T)(D^2 + \lambda)^{-(n+1)}. \tag{2.24}$$

Thus using (2.20) the integrand in (2.19) is up to a remainder,

$$(D^2 + \lambda)^{-1} X (D^2 + \lambda)^{-1} - (D^2 + \lambda)^{-1} X (D^2 + \lambda)^{-1} X (D^2 + \lambda)^{-1}$$
$$+ \cdots + (-1)^{k+1}((D^2 + \lambda)^{-1} X)^k (D^2 + \lambda)^{-1} + \cdots. \tag{2.25}$$

Using (2.24) one can move all the $(D^2 + \lambda)^{-1}$ to the right at the expense of replacing X's by $\nabla^{k_j}(X)$ and increasing the n in $(D^2 + \lambda)^{-n}$. Thus using, $(n \geq 2)$

$$\int_0^\infty (D^2 + \lambda)^{-n} d\lambda = \frac{1}{n-1} D^{2(1-n)}, \tag{2.26}$$

we get that Y is in $\Psi\mathcal{D} \cap OP^{-1}$ provided we control the remainders. To control the remainder in (2.21) one can use

$$\int_0^\infty \|(X(D^2 + \lambda)^{-1})^3\| d\lambda < \infty, \tag{2.27}$$

while the other terms are uniformly in OP^{-N} since $D^2(D^2 + \lambda)^{-1}$ is bounded by 1 in any $\mathcal{H}^s$.

To get (2.27) since $X \in OP^1$ one can replace X by $|D|$ and only integrate from $\lambda = 1$ to ∞. Then the inequality $D^2 + \lambda \geq 2|D|\lambda^{1/2}$ gives the required result. $\quad\square$

Lemma 2.3. (1) *For any N there is an element $B(t) \in \Psi\mathcal{D}$ such that modulo OP^{-N},*

$$\frac{\partial}{\partial t}(\mathrm{Log}\,(D^2 + tX) - \mathrm{Log}\,D^2 - \mathrm{Log}\,(1 + tXD^{-2})) = [D^2 + tX, B(t)]. \tag{2.28}$$

(2) *Modulo OP^{-N} one has*

$$\mathrm{Log}\,(D^2 + X) - \mathrm{Log}\,D^2 - \mathrm{Log}\,(1 + XD^{-2}) = [D^2, B_1] + [X, B_2]$$

where $B_1 = \int_0^1 B(t)\mathrm{d}t$, $B_2 = \int_0^1 tB(t)\mathrm{d}t$ are in $\Psi\mathcal{D}$.

Proof. (1) From (2.19) one has

$$\frac{\partial}{\partial t}\mathrm{Log}\,(D^2 + tX) = \int_0^\infty \frac{1}{\lambda + D^2 + tX} X \frac{1}{\lambda + D^2 + tX} d\lambda, \tag{2.29}$$

while

$$\int_0^\infty X \frac{1}{(\lambda + D^2 + tX)^2} d\lambda = X(D^2 + tX)^{-1}, \tag{2.30}$$

which is the derivative in t of $\mathrm{Log}\,(1 + tXD^{-2})$ since $X(D^2 + tX)^{-1} = XD^{-2}(1 + tXD^{-2})^{-1}$.

We thus get, calling $Z(t)$ the left hand side of (2.28),

$$Z(t) = \int_0^\infty \left[\frac{1}{\lambda + D^2 + tX}, \, X \frac{1}{\lambda + D^2 + tX} \right] d\lambda. \tag{2.31}$$

Let us define,

$$\nabla_t(T) = [D^2 + tX, T], \tag{2.32}$$

and apply the formula of Lemma 2.1(b) with $\lambda + D^2 + tX$ instead of D^2 and $T = X(\lambda + D^2 + tX)^{-1}$. We thus get,

$$\left[\frac{1}{\lambda + D^2 + tX}, T\right] = \sum_1^n (-1)^k \nabla_t^k (T(\lambda + D^2 + tX)^{-(k+1)}) + R_n, \tag{2.33}$$

where we put $(\lambda + D^2 + tX)^{-(k+1)}$ inside the argument of ∇_t^k since it is in the centralizer of ∇_t. Thus,

$$\left[\frac{1}{\lambda + D^2 + tX}, T\right] = \nabla_t \left(\sum_1^n (-1)^k \nabla_t^{k-1}(X)(\lambda + D^2 + tX)^{-(k+2)}\right) + R_n. \tag{2.34}$$

When integrated in λ the parenthesis gives,

$$B(t) = \sum_1^n (-1)^k \nabla_t^{k-1}(X) \frac{1}{k+1} \frac{1}{(D^2 + tX)^{k+1}}. \tag{2.35}$$

Let us then check that $(D^2 + tX)^{-1} \in \Psi\mathcal{D}$. We just expand it as,

$$(D^2 + tX)^{-1} = D^{-2} - D^{-2}tXD^{-2} + D^{-2}tXD^{-2}tXD^{-2} - \cdots. \tag{2.36}$$

It follows that $B(t) \in \Psi\mathcal{D}$ while,

$$Z(t) = \nabla_t(B(t)) + R_n'. \tag{2.37}$$

(2) Follows by integration using (2.35) and (2.36) to express B_j as explicit elements of $\Psi\mathcal{D} \bmod OP^{-N}$. □

2.3. The variation $\zeta_{D+A}(0) - \zeta_D(0)$

We are now ready to prove the main result of this section, we work as above with a regular spectral triple with simple dimension spectrum.

Theorem 2.4. *Let A be a gauge potential,*

(1) *The function $\zeta_{D+A}(s)$ extends to a meromorphic function with at most simple poles.*
(2) *It is regular at $s = 0$.*
(3) *One has*

$$\zeta_{D+A}(0) - \zeta_D(0) = -\!\!\!\int \mathrm{Log}\,(1 + AD^{-1}) = \sum \frac{(-1)^n}{n} \!\!\!\int (AD^{-1})^n.$$

Proof. (1) We start from the expansional formula

$$\mathrm{e}^{A+B}\mathrm{e}^{-A} = \sum_0^\infty \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} B(t_1)B(t_2)\ldots B(t_n) \prod \mathrm{d}t_i \tag{2.38}$$

where

$$B(t) = \mathrm{e}^{tA}B\mathrm{e}^{-tA}. \tag{2.39}$$

We take $A = -\frac{s}{2}\mathrm{Log}\,D^2$ and $B = -\frac{s}{2}Y$ so that,

$$\mathrm{e}^{A+B} = (D^2 + X)^{-s/2}, \quad \mathrm{e}^A = (D^2)^{-s/2}. \tag{2.40}$$

We define the one parameter group,

$$\sigma_u(T) = (D^2)^{u/2}T(D^2)^{-u/2}, \tag{2.41}$$

so that with the above notations we get,

$$B(t) = -\frac{s}{2}\sigma_{-st}(Y). \tag{2.42}$$

We can thus write,

$$
(D^2 + X)^{-s/2} = (D^2)^{-s/2} + \sum_{n=1}^{\infty} \left(-\frac{s}{2}\right)^n
$$

$$
\times \left(\int_{0 \le t_1 \le \cdots \le t_n \le 1} \sigma_{-st_1}(Y) \ldots \sigma_{-st_i}(Y) \ldots \sigma_{-st_n}(Y) \prod dt_i \right) (D^2)^{-s/2}. \tag{2.43}
$$

Since by Lemma 2.2 one has $Y \in \Psi\mathcal{D} \cap OP^{-1}$ for any given half plane $H = \{z; \Re(z) \ge a\}$ only finitely many terms of the sum (2.43) contribute to the singularities in H of the function $\zeta_{D+A}(s) = \mathrm{Tr}((D^2 + X)^{-s/2})$ and the expansion of the one parameter group σ_u (*cf.* [6])

$$
\sigma_{2z}(T) = T + z\epsilon(T) + \frac{z(z-1)}{2!}\epsilon^2(T) + \cdots + \frac{z(z-1)\cdots(z-n+1)}{n!}\epsilon^n(T) \quad \mod OP^{q-(n+1)} \tag{2.44}
$$

where $T \in OP^q$ and,

$$
\epsilon(T) = [D^2, T]D^{-2} = [D^2, TD^{-2}] \tag{2.45}
$$

gives the required meromorphic continuation.

(2) By hypothesis the functions of the form $\mathrm{Tr}(P|D|^{-s})$ for $P \in \Psi\mathcal{D}$ have at most simple poles thus only the first term of the infinite sum in (2.43) can contribute to the value $\zeta_{D+A}(0) - \zeta_D(0)$. This first term is

$$
-\frac{s}{2} \int_0^1 \sigma_{-st}(Y) dt (D^2)^{-s/2},
$$

and using (2.44) one can replace $\sigma_{-st}(Y)$ by Y without altering the value of $\zeta_{D+A}(0) - \zeta_D(0)$ which is hence, using the definition of the residue

$$
\fint P = \mathrm{Res}_{s=0}\mathrm{Tr}(P|D|^{-s}), \tag{2.46}
$$

given by

$$
\zeta_{D+A}(0) - \zeta_D(0) = -\frac{1}{2}\fint Y = -\frac{1}{2}\fint \mathrm{Log}\,(1 + XD^{-2}), \tag{2.47}
$$

using Lemma 2.3 (2) and the trace property (1.4).

(3) For any elements $a, b \in \Psi\mathcal{D} \cap OP^{-1}$ one has the identity

$$
\fint \mathrm{Log}\,((1 + a)(1 + b)) = \fint \mathrm{Log}\,(1 + a) + \fint \mathrm{Log}\,(1 + b). \tag{2.48}
$$

This can be checked directly using the expansion

$$
\mathrm{Log}\,(1 + a) = \sum_1^{\infty}(-1)^{n+1}\frac{a^n}{n},
$$

and the trace property (1.4) of the residue. In fact one can reduce it to the identity

$$
\fint (t + b)^{-1}(t + a)^{-1}(2t + a + b) = \fint ((t + a)^{-1} + (t + b)^{-1}),
$$

which follows from (1.4) and the equality

$$
(t + a)^{-1}(2t + a + b)(t + b)^{-1} = (t + a)^{-1} + (t + b)^{-1}.
$$

Applying (2.48) to $a = D^{-1}A$ and $b = AD^{-1}$ one gets, with $X = DA + AD + A^2$ as above,

$$
\fint \mathrm{Log}\,(1 + XD^{-2}) = 2\fint \mathrm{Log}\,(1 + AD^{-1}), \tag{2.49}
$$

which combined with (2.47) gives the required equality. $\quad\square$

3. Yang–Mills + Chern–Simons

We work in dimension ≤ 4 and make the following hypothesis of *vanishing tadpole* (*cf.* Fig. 4)

$$\fint a[D, b]D^{-1} = 0, \quad \forall a, b \in \mathcal{A}. \tag{3.1}$$

By Theorem 2.4 this condition is equivalent to the vanishing of the first order variation of the (scale independent part of) the spectral action under inner fluctuations, and is thus a natural hypothesis.

Given a Hochschild cochain φ of dimension n on an algebra $\mathcal{A}$, normalized so that

$$\varphi(a_0, a_1, \ldots, a_n) = 0,$$

if any of the a_j for $j > 0$ is a scalar, it defines (*cf.* [3]) a functional on the universal n-forms $\Omega^n(\mathcal{A})$ by the equality

$$\int_\varphi a_0 \mathrm{d}a_1 \cdots \mathrm{d}a_n = \varphi(a_0, a_1, \ldots, a_n). \tag{3.2}$$

When φ is a Hochschild cocycle one has

$$\int_\varphi a\omega = \int_\varphi \omega a, \quad \forall a \in \mathcal{A}. \tag{3.3}$$

The boundary operator B_0 defined on normalized cochains by

$$(B_0\varphi)(a_0, a_1, \ldots, a_{n-1}) = \varphi(1, a_0, a_1, \ldots, a_{n-1}), \tag{3.4}$$

is defined in such a way that

$$\int_\varphi \mathrm{d}\omega = \int_{B_0\varphi} \omega. \tag{3.5}$$

Working in dimension ≤ 4 means that

$$D^{-1} \in \mathcal{L}^{(4,\infty)}, \tag{3.6}$$

i.e. that D^{-1} is an infinitesimal of order $\frac{1}{4}$ (*cf.* [3]). The following functional is then a Hochschild cocycle and is given as Dixmier trace of infinitesimals of order one,

$$\tau_0(a^0, a^1, a^2, a^3, a^4) = \fint a^0[D, a^1]D^{-1}[D, a^2]D^{-1}[D, a^3]D^{-1}[D, a^4]D^{-1}. \tag{3.7}$$

The following functional uses the residue in an essential manner,

$$\varphi(a^0, a^1, a^2, a^3) = \fint a^0[D, a^1]D^{-1}[D, a^2]D^{-1}[D, a^3]D^{-1}. \tag{3.8}$$

Lemma 3.1. (1) $b\varphi = -\tau_0$
(2) $bB_0\tau_0 = 2\tau_0$
(3) $B_0\varphi = 0$.

Proof. (1) One has,

$$b\varphi(a^0, \ldots, a^4) = \fint a^0 a^1[D, a^2]D^{-1}[D, a^3]D^{-1}[D, a^4]D^{-1}$$

$$- \fint a^0(a^1[D, a^2] + [D, a^1]a^2)D^{-1}[D, a^3]D^{-1}[D, a^4]D^{-1}$$

$$+ \fint a^0[D, a^1]D^{-1}(a^2[D, a^3] + [D, a^2]a^3)D^{-1}[D, a^4]D^{-1}$$

$$- \int a^0[D,a^1]D^{-1}[D,a^2]D^{-1}(a^3[D,a^4] + [D,a^3]a^4)D^{-1}$$

$$+ \int a^4 a^0[D,a^1]D^{-1}[D,a^2]D^{-1}[D,a^3]D^{-1}.$$

Thus using

$$aD^{-1} - D^{-1}a = D^{-1}[D,a]D^{-1}, \tag{3.9}$$

we get (2).

(2) One has

$$B_0 \tau_0(a^0, a^1, a^2, a^3) = \int [D,a^0]D^{-1}[D,a^1]D^{-1}[D,a^2]D^{-1}[D,a^3]D^{-1}$$

$$= -\varphi(a^0, a^1, a^2, a^3) + \tilde{\varphi}(a^0, a^1, a^2, a^3),$$

where

$$\tilde{\varphi}(a^0, a^1, a^2, a^3) = \int a^0 D^{-1}[D,a^1]D^{-1}[D,a^2]D^{-1}[D,a^3].$$

Thus it is enough to check that $b\tilde{\varphi} = \tau_0$. One has

$$b\tilde{\varphi}(a^0, \ldots, a^4) = \int a^0 a^1 D^{-1}[D,a^2]D^{-1}[D,a^3]D^{-1}[D,a^4]$$

$$- \int a^0 D^{-1}(a^1[D,a^2] + [D,a^1]a^2)D^{-1}[D,a^3]D^{-1}[D,a^4]$$

$$+ \int a^0 D^{-1}[D,a^1]D^{-1}(a^2[D,a^3] + [D,a^2]a^3)D^{-1}[D,a^4]$$

$$- \int a^0 D^{-1}[D,a^1]D^{-1}[D,a^2]D^{-1}(a^3[D,a^4] + [D,a^3]a^4)$$

$$+ \int a^4 a^0 D^{-1}[D,a^1]D^{-1}[D,a^2]D^{-1}[D,a^3]$$

$$= \int a^0 D^{-1}[D,a^1]D^{-1}[D,a^2]D^{-1}[D,a^3]D^{-1}[D,a^4]$$

and using (3.9) one gets the required equality since, using (3.6),

$$\int a^0[D,a^1]D^{-1}[D,a^2]D^{-1}[D,a^3]D^{-1}[D,a^4]D^{-1} = \int a^0 D^{-1}[D,a^1]D^{-1}[D,a^2]D^{-1}[D,a^3]D^{-1}[D,a^4].$$

(3) We use the notation

$$\alpha(a) = DaD^{-1}, \quad \forall a \in \mathcal{A}. \tag{3.10}$$

Note that in general $\alpha(a) \notin \mathcal{A}$. One has

$$\alpha(ab) = \alpha(a)\alpha(b), \quad \forall a, b \in \mathcal{A}.$$

Let us show that the tadpole hypothesis (3.1) implies that for any three elements $a, b, c \in \mathcal{A}$,

$$\int \alpha^{\epsilon_1}(a)\alpha^{\epsilon_2}(b)\alpha^{\epsilon_3}(c) = \int abc, \tag{3.11}$$

for all $\epsilon_j \in \{0, 1\}$. The trace property of the residue shows that this holds when all $\epsilon_j = 1$. One is thus reduced to show that

$$\int \alpha(x)y = \int xy, \quad \forall x, y \in \mathcal{A},$$

which follows from (3.1). One has by construction

$$B_0\varphi(a_0, a_1, a_2) = \fint (\alpha(a_0) - a_0)(\alpha(a_1) - a_1)(\alpha(a_2) - a_2),$$

which vanishes since the terms cancel pairwise.   □

Lemma 3.2. *One has for any $A \in \Omega^1$ the equality*

$$\fint AD^{-1}AD^{-1} = -\int_\varphi A\mathrm{d}A.$$

Proof. Let us first show that for any $a_j \in \mathcal{A}$ one has

$$\fint a_0[D, a_1]D^{-1}a_2[D, a_3]D^{-1} = -\varphi(a^0, a^1, a^2, a^3). \tag{3.12}$$

It suffices using (3.9) to show that

$$\fint a_0[D, a_1]a_2 D^{-1}[D, a_3]D^{-1} = 0,$$

which follows using

$$a_0[D, a_1]a_2 = a_0[D, a_1 a_2] - a_0 a_1[D, a_2],$$

and the vanishing of

$$\fint a[D, b]D^{-1}[D, c]D^{-1} = \fint a(\alpha(b) - b)(\alpha(c) - c) = 0, \quad \forall a, b, c \in \mathcal{A}$$

using (3.7). Let then $A_1 = a_0\mathrm{d}a_1$, $A_2 = a_2\mathrm{d}a_3$, one has

$$\fint A_1 D^{-1} A_2 D^{-1} = -\int_\varphi A_1 \mathrm{d}A_2,$$

since $\mathrm{d}A_2 = \mathrm{d}a_2\mathrm{d}a_3$, and the same holds for any $A_j \in \Omega^1$ so that Lemma 3.2 follows.   □

Lemma 3.3. *One has for any $A \in \Omega^1$ the equality*

$$\fint (AD^{-1})^4 = \int_{\tau_0} A^4.$$

Proof. It is enough to check that with a_j, b_j in $\mathcal{A}$ one has

$$\int_{\tau_0} a_1\mathrm{d}b_1 a_2\mathrm{d}b_2 a_3\mathrm{d}b_3 a_4\mathrm{d}b_4 = \fint A_1 D^{-1} A_2 D^{-1} A_3 D^{-1} A_4 D^{-1}, \quad A_j = a_j[D, b_j].$$

Since there are 4 terms D^{-1} one is in the domain of the Dixmier trace and one can freely permute the factors D^{-1} with the elements of $\mathcal{A}$ in computing the residue of the right hand side. One can thus assume that $a_2 = a_3 = a_4 = 1$. The result then follows from (3.7).   □

Lemma 3.4. *One has for any $A_j \in \Omega^1$ the equality*

$$\fint A_1 D^{-1} A_2 D^{-1} A_3 D^{-1} = \int_{\varphi + \frac{1}{2}B_0\tau_0} A_1 A_2 A_3 - \frac{1}{2}\left(\int_{\tau_0} (\mathrm{d}A_1)A_2 A_3 + \int_{\tau_0} A_1\mathrm{d}A_2 A_3 + \int_{\tau_0} A_1 A_2\mathrm{d}A_3\right). \tag{3.13}$$

Proof. We can take $A_j = a_j \mathrm{d}b_j$ and the first task is to reorder

$$
\begin{aligned}
a_1 \mathrm{d}b_1 a_2 \mathrm{d}b_2 a_3 \mathrm{d}b_3 &= a_1 \mathrm{d}b_1 a_2 (\mathrm{d}(b_2 a_3) - b_2 \mathrm{d}a_3) \mathrm{d}b_3 \\
&= a_1 (\mathrm{d}(b_1 a_2) - b_1 \mathrm{d}a_2) \mathrm{d}(b_2 a_3) \mathrm{d}b_3 - a_1 (\mathrm{d}(b_1 a_2 b_2) - b_1 \mathrm{d}(a_2 b_2)) \mathrm{d}a_3 \mathrm{d}b_3 \\
&= a_1 \mathrm{d}(b_1 a_2) \mathrm{d}(b_2 a_3) \mathrm{d}b_3 - a_1 b_1 \mathrm{d}a_2 \mathrm{d}(b_2 a_3) \mathrm{d}b_3 \\
&\quad - a_1 \mathrm{d}(b_1 a_2 b_2) \mathrm{d}a_3 \mathrm{d}b_3 + a_1 b_1 \mathrm{d}(a_2 b_2) \mathrm{d}a_3 \mathrm{d}b_3.
\end{aligned}
\tag{3.14}
$$

We thus get

$$
\begin{aligned}
\int_\varphi A_1 A_2 A_3 &= \fint a_1 [D, b_1 a_2] D^{-1} [D, b_2 a_3] D^{-1} [D, b_3] D^{-1} \\
&\quad - \fint a_1 b_1 [D, a_2] D^{-1} [D, b_2 a_3] D^{-1} [D, b_3] D^{-1} \\
&\quad - \fint a_1 [D, b_1 a_2 b_2] D^{-1} [D, a_3] D^{-1} [D, b_3] D^{-1} \\
&\quad + \fint a_1 b_1 [D, a_2 b_2] D^{-1} [D, a_3] D^{-1} [D, b_3] D^{-1} \\
&= \fint a_1 [D, b_1] a_2 D^{-1} [D, b_2 a_3] D^{-1} [D, b_3] D^{-1} \\
&\quad - \fint a_1 [D, b_1] a_2 b_2 D^{-1} [D, a_3] D^{-1} [D, b_3] D^{-1}.
\end{aligned}
$$

Using $[D^{-1}, b_2] = -D^{-1}[D, b_2] D^{-1}$ we thus get,

$$
\begin{aligned}
\int_\varphi A_1 A_2 A_3 &= \fint a_1 [D, b_1] a_2 D^{-1} [D, b_2] a_3 D^{-1} [D, b_3] D^{-1} \\
&\quad - \fint a_1 [D, b_1] a_2 D^{-1} [D, b_2] D^{-1} [D, a_3] D^{-1} [D, b_3] D^{-1}.
\end{aligned}
\tag{3.15}
$$

Next one has using (3.14)

$$
\begin{aligned}
\int_{B_0 \tau_0} A_1 A_2 A_3 &= \fint [D, a_1] D^{-1} [D, b_1 a_2] D^{-1} [D, b_2 a_3] D^{-1} [D, b_3] D^{-1} \\
&\quad - \fint [D, a_1 b_1] D^{-1} [D, a_2] D^{-1} [D, b_2 a_3] D^{-1} [D, b_3] D^{-1} \\
&\quad - \fint [D, a_1] D^{-1} [D, b_1 a_2 b_2] D^{-1} [D, a_3] D^{-1} [D, b_3] D^{-1} \\
&\quad + \fint [D, a_1 b_1] D^{-1} [D, a_2 b_2] D^{-1} [D, a_3] D^{-1} [D, b_3] D^{-1}.
\end{aligned}
$$

Since one is in the domain of the Dixmier trace, one can permute D^{-1} with a for $a \in \mathcal{A}$. Thus the first two terms combine to give,

$$
\begin{aligned}
&\fint [D, a_1] D^{-1} [D, b_1] a_2 D^{-1} [D, b_2 a_3] D^{-1} [D, b_3] D^{-1} \\
&\quad - \fint a_1 [D, b_1] D^{-1} [D, a_2] D^{-1} [D, b_2 a_3] D^{-1} [D, b_3] D^{-1},
\end{aligned}
$$

and the last two terms combine to give,

$$
\begin{aligned}
&\fint a_1 [D, b_1] D^{-1} [D, a_2 b_2] D^{-1} [D, a_3] D^{-1} [D, b_3] D^{-1} \\
&\quad - \fint [D, a_1] D^{-1} [D, b_1] a_2 b_2 D^{-1} [D, a_3] D^{-1} [D, b_3] D^{-1}.
\end{aligned}
$$

Thus these 4 terms add up to give

$$\int_{B_0\tau_0} A_1 A_2 A_3 = \fint [D, a_1] D^{-1} [D, b_1] a_2 D^{-1} [D, b_2] a_3 D^{-1} [D, b_3] D^{-1}$$
$$- \fint a_1 [D, b_1] D^{-1} [D, a_2] D^{-1} [D, b_2] a_3 D^{-1} [D, b_3] D^{-1}$$
$$+ \fint a_1 [D, b_1] D^{-1} a_2 [D, b_2] D^{-1} [D, a_3] D^{-1} [D, b_3] D^{-1}. \tag{3.16}$$

Combining this with (3.15) thus gives,

$$\int_{\varphi + \frac{1}{2} B_0 \tau_0} A_1 A_2 A_3 = \fint a_1 [D, b_1] a_2 D^{-1} [D, b_2] a_3 D^{-1} [D, b_3] D^{-1}$$
$$+ \frac{1}{2} \fint [D, a_1] D^{-1} [D, b_1] D^{-1} a_2 [D, b_2] D^{-1} a_3 [D, b_3] D^{-1}$$
$$- \frac{1}{2} \fint a_1 [D, b_1] D^{-1} [D, a_2] D^{-1} [D, b_2] D^{-1} a_3 [D, b_3] D^{-1}$$
$$- \frac{1}{2} \fint a_1 [D, b_1] D^{-1} a_2 [D, b_2] D^{-1} [D, a_3] D^{-1} [D, b_3] D^{-1}. \tag{3.17}$$

But one has, using $[a, D^{-1}] = D^{-1} [D, a] D^{-1}$,

$$\fint a_1 [D, b_1] a_2 D^{-1} [D, b_2] a_3 D^{-1} [D, b_3] D^{-1}$$
$$= \fint a_1 [D, b_1] D^{-1} [D, a_2] D^{-1} [D, b_2] a_3 D^{-1} [D, b_3] D^{-1}$$
$$+ \fint a_1 [D, b_1] D^{-1} a_2 [D, b_2] a_3 D^{-1} [D, b_3] D^{-1}$$
$$= \fint a_1 [D, b_1] D^{-1} [D, a_2] D^{-1} [D, b_2] D^{-1} a_3 [D, b_3] D^{-1}$$
$$+ \fint a_1 [D, b_1] D^{-1} a_2 [D, b_2] D^{-1} [D, a_3] D^{-1} [D, b_3] D^{-1}$$
$$+ \fint A_1 D^{-1} A_2 D^{-1} A_3 D^{-1},$$

which combined with (3.17) gives the required equality. □

We can now state the main result

Theorem 3.5. *Under the tadpole hypothesis* (3.1) *one has*
(1) $\psi = \varphi + \frac{1}{2} B_0 \tau_0$ *is a cyclic 3-cocycle given (with $\alpha(x) = DxD^{-1}$) by*

$$\psi(a_0, a_1, a_2, a_3) = \frac{1}{2} \fint (\alpha(a_0) a_1 \alpha(a_2) a_3 - a_0 \alpha(a_1) a_2 \alpha(a_3)) \tag{3.18}$$

(2) *For any $A \in \Omega^1$ one has*

$$\fint \mathrm{Log} (1 + A D^{-1}) = -\frac{1}{4} \int_{\tau_0} (\mathrm{d}A + A^2)^2 + \frac{1}{2} \int_{\psi} \left(A \mathrm{d}A + \frac{2}{3} A^3 \right). \tag{3.19}$$

Proof. (1) By Lemma 3.1 ψ is a Hochschild cocycle. Moreover by Lemma 3.1 it is in the kernel of B_0 and is hence cyclic. Expanding the expression

$$\psi(a_0, a_1, a_2, a_3) = \frac{1}{2} \fint (\alpha(a_0) + a_0)(\alpha(a_1) - a_1)(\alpha(a_2) - a_2)(\alpha(a_3) - a_3),$$

and using (3.11), one gets (3.18).

(2) One has

$$\fint \mathrm{Log}\,(1 + AD^{-1}) = -\frac{1}{2}\,\fint (AD^{-1})^2 + \frac{1}{3}\,\fint (AD^{-1})^3 - \frac{1}{4}\,\fint (AD^{-1})^4.$$

Both sides of (3.19) are thus polynomials in A and it is enough to compare the monomials of degree 2, 3 and 4. In degree 2 the right hand side of (3.19) gives

$$-\frac{1}{4}\int_{\tau_0} (\mathrm{d}A)^2 + \frac{1}{2}\int_\psi A\mathrm{d}A = \frac{1}{2}\int_\varphi A\mathrm{d}A,$$

using (3.5). Thus by Lemma 3.2 one gets the same as the term of degree two in the left hand side of (3.19). In degree 4 the right hand side of (3.19) gives

$$-\frac{1}{4}\int_{\tau_0} A^4 = -\frac{1}{4}\,\fint (AD^{-1})^4,$$

by Lemma 3.3. It remains to handle the cubic terms, the right hand side of (3.19) gives

$$-\frac{1}{4}\int_{\tau_0} (\mathrm{d}AA^2 + A^2\mathrm{d}A) + \frac{1}{3}\int_\psi A^3,$$

which using Lemma 3.4 gives

$$\frac{1}{3}\,\fint (AD^{-1})^3 + \frac{1}{6}\int_{\tau_0} (\mathrm{d}AA^2 + A\mathrm{d}AA + A^2\mathrm{d}A) - \frac{1}{4}\int_{\tau_0} (\mathrm{d}AA^2 + A^2\mathrm{d}A).$$

Thus it remains to show that the sum of the last two terms is zero. In fact

$$\int_{\tau_0} \mathrm{d}AA^2 = \int_{\tau_0} A\mathrm{d}AA = \int_{\tau_0} A^2\mathrm{d}A.$$

This follows from the more general equality

$$\int_{\tau_0} \omega_1\omega_2\omega_3\omega_4 = \int_{\tau_0} \omega_2\omega_3\omega_4\omega_1, \quad \forall\omega_j \in \Omega^1, \tag{3.20}$$

which is seen as follows. Let $\omega_j = a_j\mathrm{d}b_j$, then

$$\int_{\tau_0} \omega_1\omega_2\omega_3\omega_4 = \fint a_1[D, b_1]D^{-1}a_2[D, b_2]D^{-1}a_3[D, b_3]D^{-1}a_4[D, b_4]D^{-1},$$

so that (3.20) follows from the trace property of the residue. □

Combining this result with Theorem 2.4 one gets

Corollary 3.6. *The variation under inner fluctuations of the scale independent terms of the spectral action is given in dimension 4 by*

$$\zeta_{D+A}(0) - \zeta_D(0) = \frac{1}{4}\int_{\tau_0} (\mathrm{d}A + A^2)^2 - \frac{1}{2}\int_\psi \left(A\mathrm{d}A + \frac{2}{3}A^3\right).$$

Note that there is still some freedom in the choice of the cocycles τ_0 and ψ involved in Theorem 3.5. Indeed let $B = AB_0$ be the fundamental boundary operator in cyclic cohomology [3], one has

Proposition 3.7. (1) *Theorem 3.5 still holds after the replacements $\tau_0 \to \tau_0 + \rho$ and $\psi \to \psi + \frac{1}{2}B_0\rho$ for any Hochschild 4-cocycle ρ such that $B_0\rho$ is already cyclic i.e. such that $AB_0\rho = 4B_0\rho$.*
(2) *If the cocycle ψ is in the image of B i.e. if $\psi \in B(Z^4(\mathcal{A}, \mathcal{A}^*))$ one can eliminate ψ by a redefinition of τ_0.*

Proof. (1) We first show that $\int_\rho$ is a graded trace (*cf.* [3], Chapter III, Lemma 18). First since ρ is a Hochschild cocycle one has

$$\int_\rho a\omega = \int_\rho \omega a, \quad \forall a \in \mathcal{A}.$$

To show that $\int_\rho$ is a graded trace it is enough to check that

$$\int_\rho \mathrm{d}a(a_0\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3) = -\int_\rho a_0\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3\mathrm{d}a,$$

i.e. that

$$B_0\rho(aa_0, \ldots, a_3) - \rho(a, a_0, \ldots, a_3) = -\rho(a_0, \ldots, a_3, a),$$

which follows (*cf.* [3], Chapter III, Lemma 18) from

$$B_0b + b'B_0 = \mathrm{id} - \lambda,$$

(where λ is the cyclic permutation) and $b\rho = 0$, $bB_0\rho = 0$. We need to show that the right hand side of (3.19) is unaltered by the above replacements. For the terms of degree 4 one has to show that

$$\int_\rho A^4 = 0,$$

which holds because $\int_\rho$ is a graded trace. For the terms of degree 3 one has

$$\int_\rho (\mathrm{d}AA^2 + A^2\mathrm{d}A) - \frac{4}{3}\int_{\frac{1}{2}B_0\rho} A^3 = \int_\rho \left(\mathrm{d}AA^2 + A^2\mathrm{d}A - \frac{2}{3}\mathrm{d}(A^3)\right),$$

and the graded trace property of $\int_\rho$ shows that this vanishes. For the quadratic terms one has

$$\int_\rho (\mathrm{d}A)^2 - 2\int_{\frac{1}{2}B_0\rho} A\mathrm{d}A = \int_\rho ((\mathrm{d}A)^2 - \mathrm{d}(A\mathrm{d}A)) = 0.$$

(2) By [3] Chapter III, Lemma 19, the condition $\psi \in B(Z^4(\mathcal{A}, \mathcal{A}^*))$ implies that one can find a Hochschild 4-cocycle ρ such that $B_0\rho$ is already cyclic and equal to -2ψ thus using (1) one can eliminate ψ. $\quad\square$

The above ambiguity can thus be written in the form

$$\psi \to \psi + \delta, \quad \forall \delta \in B(Z^4(\mathcal{A}, \mathcal{A}^*)) \tag{3.21}$$

and it does not alter the periodic cyclic cohomology class of the three cocycle ψ.

The Yang–Mills action given by

$$YM_\tau(A) = \int_\tau (\mathrm{d}A + A^2)^2,$$

is automatically gauge invariant under the gauge transformations

$$A \to \gamma_u(A) = u\mathrm{d}u^* + uAu^*, \quad \forall u \in \mathcal{A}, uu^* = u^*u = 1, \tag{3.22}$$

as soon as τ is a Hochschild cocycle since $F(A) = \mathrm{d}A + A^2$ transforms covariantly *i.e.* $F(\gamma_u(A)) = uF(A)u^*$. This action and its precise relation with the usual Yang–Mills functional is discussed at length in [3] Chapter VI.

We now discuss briefly the invariance of the Chern–Simons action. An early instance of this action in terms of cyclic cohomology can be found in [9]. It is not in general invariant under gauge transformations but one has the following more subtle invariance,

Proposition 3.8. *Let ψ be a cyclic three cocycle on $\mathcal{A}$. The functional*

$$CS_\psi(A) = \int_\psi A\mathrm{d}A + \frac{2}{3}A^3$$

fulfills the following invariance rule under the gauge transformation $\gamma_u(A) = u\mathrm{d}u^* + uAu^*$,

$$CS_\psi(\gamma_u(A)) = CS_\psi(A) + \frac{1}{3}\langle\psi, u\rangle$$

where $\langle\psi, u\rangle$ is the pairing between $HC^3(\mathcal{A})$ and $K_1(\mathcal{A})$.

Proof. Let $A' = \gamma_u(A) = u \mathrm{d}u^* + uAu^*$. One has

$$\mathrm{d}A' = \mathrm{d}u\mathrm{d}u^* + \mathrm{d}uAu^* + u\mathrm{d}Au^* - uA\mathrm{d}u^*,$$
$$A'\mathrm{d}A' = u\mathrm{d}u^*\mathrm{d}u\mathrm{d}u^* + u\mathrm{d}u^*\mathrm{d}uAu^* + u\mathrm{d}u^*u\mathrm{d}Au^* - u\mathrm{d}u^*uA\mathrm{d}u^*$$
$$+ uAu^*\mathrm{d}u\mathrm{d}u^* + uAu^*\mathrm{d}uAu^* + uA\mathrm{d}Au^* - uA^2\mathrm{d}u^*.$$

So that using the graded trace property of $\int_\psi$ one gets

$$\int_\psi (A'\mathrm{d}A' - A\mathrm{d}A) = \int_\psi (u\mathrm{d}u^*\mathrm{d}u\mathrm{d}u^* + \mathrm{d}u^*\mathrm{d}uA - \mathrm{d}u^*u\mathrm{d}u^*uA + u^*\mathrm{d}u\mathrm{d}u^*uA$$
$$+ \mathrm{d}u^*u\mathrm{d}A + u^*\mathrm{d}uA^2 - \mathrm{d}u^*uA^2),$$

which using

$$\int_\psi \mathrm{d}u^*u\mathrm{d}A = - \int_\psi \mathrm{d}u^*\mathrm{d}uA,$$

gives

$$\int_\psi (A'\mathrm{d}A' - A\mathrm{d}A) = \int_\psi (u\mathrm{d}u^*\mathrm{d}u\mathrm{d}u^* + 2u^*\mathrm{d}u\mathrm{d}u^*uA + 2u^*\mathrm{d}uA^2).$$

Next one has

$$\int_\psi (A'^3 - A^3) = \int_\psi ((u\mathrm{d}u^*)^3 + 3(u\mathrm{d}u^*)^2uAu^* + 3u\mathrm{d}u^*uA^2u^*).$$

Since $\mathrm{d}u^*u = -u^*\mathrm{d}u$, the terms in A^2 cancel in the variation of CS_ψ. Similarly one has $\mathrm{d}u^*u\mathrm{d}u^*u = -u^*\mathrm{d}u\mathrm{d}u^*u$ so that the terms in A also cancel. One thus obtains

$$CS_\psi(\gamma_u(A)) - CS_\psi(A) = \int_\psi \left(u\mathrm{d}u^*\mathrm{d}u\mathrm{d}u^* + \frac{2}{3}(u\mathrm{d}u^*)^3 \right).$$

One has $(u\mathrm{d}u^*)^3 = -u\mathrm{d}u^*\mathrm{d}u\mathrm{d}u^*$ which gives the required result. $\quad\square$

Corollary 3.9. *Let ψ be the cyclic three cocycle of Theorem 3.5 then its pairing with the K_1-group vanishes identically,*

$$\langle \psi, u \rangle = 0, \quad \forall u \in K_1(\mathcal{A}).$$

Proof. The effect of the gauge transformation (3.22) is to replace the operator $D + A$ by the unitarily equivalent operator $D + \gamma_u(A) = u(D + A)u^*$, thus the spectral invariants are unaltered by such a transformation. Since the Yang–Mills term

$$\frac{1}{4} \int_{\tau_0} (\mathrm{d}A + A^2)^2,$$

is invariant under gauge transformations, it follows that so is the Chern–Simons term which implies by Proposition 3.8 that the pairing between the cyclic cocycle ψ and the unitary u is zero. Tensoring the original spectral triple by the finite geometry $(M_n(\mathbb{C}), \mathbb{C}^n, 0)$ allows us to apply the same argument to unitaries in $M_n(\mathcal{A})$ and shows that the pairing with the K_1-group vanishes identically. $\quad\square$

As an example where Proposition 3.7 applies we consider the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ associated to a spin Riemannian four manifold.

Lemma 3.10. *Let $(\mathcal{A}, \mathcal{H}, D)$ be the spectral triple associated to a compact spin Riemannian four manifold M. Then one has, with φ given by (3.8)*

$$\varphi = \frac{1}{6} \fint ([D^2, a^0][D, a^1] - [D, a^0][D^2, a^1])([D^2, a^2][D, a^3] - [D, a^2][D^2, a^3])D^{-6}. \tag{3.23}$$

Proof. The right hand side of (3.23) only depends on the principal symbols of the operators $[D, a^j]$ and $[D^2, a^j]$ which are given respectively by $\frac{1}{i}\partial_\mu a^j \gamma^\mu$ and $-2i\partial_\mu a^j g^{\mu\lambda}\xi_\lambda$. Thus the principal symbol of $[D^2, a^0][D, a^1] - [D, a^0][D^2, a^1]$ is

$$-2(\partial_\mu a^0 \xi^\mu \partial_\nu a^1 \gamma^\nu - \partial_\nu a^0 \gamma^\nu \partial_\mu a^1 \xi^\mu).$$

The integral over the unit sphere bundle of the product of two symbols of the form $\gamma^\nu \xi_\mu$ is zero unless the indices are the same and one gets, for the normalized integral,

$$\int_{S^*} \mathrm{Tr}((\gamma^\nu \xi^\mu)(\gamma^\kappa \xi^\lambda))\mathrm{d}v = g^{\mu\lambda} g^{\nu\kappa}$$

since the dimension of spinors is equal to 4. Let $\lambda > 0$ be the constant such that, for any pseudodifferential operator P with total symbol σ one has

$$\fint P = \lambda \int_{S^*} \mathrm{Tr}(\sigma_4(P))\mathrm{d}v\mathrm{d}x.$$

We thus get the following formula for the right hand side of (3.23),

$$\frac{2}{3}\lambda \int_M g^{\mu\lambda} g^{\nu\kappa} (\partial_\mu a^0 \partial_\nu a^1 - \partial_\nu a^0 \partial_\mu a^1)(\partial_\lambda a^2 \partial_\kappa a^3 - \partial_\kappa a^3 \partial_\lambda a^2)\mathrm{d}x = \frac{4}{3}\lambda \langle \mathrm{d}a^0 \wedge \mathrm{d}a^1, \mathrm{d}a^2 \wedge \mathrm{d}a^3 \rangle$$

where the inner product between two forms is given by

$$\langle \omega_1, \omega_2 \rangle = \int_M \omega_1 \wedge \star\omega_2.$$

Let us show that the left hand side of (3.23) is equal to

$$\varphi = \frac{4}{3}\lambda \int_M a^0 \mathrm{d}a^1 \wedge \mathrm{d}(\star(\mathrm{d}a^2 \wedge \mathrm{d}a^3)) \tag{3.24}$$

The equality (3.23) then follows from Stokes' formula since M has no boundary. One has by definition

$$\varphi(a^0, a^1, a^2, a^3) = \fint a^0 [D, a^1] D^{-1}[D, a^2] D^{-1}[D, a^3] D^{-1},$$

and to prove (3.24) we can take normal coordinates at $x \in M$ and compute the subprincipal symbol of $D^{-1}[D, a^2]D^{-1}[D, a^3]D^{-1}$. We recall that the symbol of a product of pseudodifferential operators P_j with total symbols σ_j is given by

$$\sigma(P_1 P_2) = \sigma_1 \sigma_2 + \frac{1}{i} \frac{\partial \sigma_1}{\partial \xi_\mu} \frac{\partial \sigma_2}{\partial x_\mu} + \cdots.$$

In particular with $\sigma(D) = \xi_\mu \gamma^\mu$, this is coherent with $[D, a] = \frac{1}{i}\partial_\mu a \gamma^\mu$. The symbol of D^{-1} is up to operators of order -3 given by

$$\sigma(D^{-1}) = \|\xi\|^{-2}\xi_\mu \gamma^\mu$$

since we are in normal coordinates. Since there are always four factors of $\frac{1}{i}$ one can neglect them and just compute

$$\sigma = \frac{\partial \sigma_1}{\partial \xi_\mu} \partial_\mu \sigma_2 \sigma_3 + \frac{\partial \sigma_1}{\partial \xi_\mu} \sigma_2 \partial_\mu \sigma_3 + \sigma_1 \frac{\partial \sigma_2}{\partial \xi_\mu} \partial_\mu \sigma_3$$

where $\sigma_j = \partial_k a^j \|\xi\|^{-2}\xi_\ell \gamma^k \gamma^\ell$. To compute the residue one uses

$$\int_{S^*} \xi_\mu^2 \mathrm{d}v = \frac{1}{4}, \quad \int_{S^*} \xi_\mu^4 \mathrm{d}v = \frac{1}{8}, \quad \int_{S^*} \xi_\mu^2 \xi_\nu^2 \mathrm{d}v = \frac{1}{24},$$

which gives

$$\int_{S^*} \xi_i \xi_j \frac{\partial}{\partial \xi_k}(\|\xi\|^{-2}\xi_\ell)\mathrm{d}v = \frac{1}{12}(2\delta_{ij}\delta_{k\ell} - \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}).$$

Note the identity

$$\sum_\mu \mathrm{Tr}(\gamma^a \gamma^\mu \gamma^a \gamma^\mu) = -8$$

which allows us to check that the coefficient of a term like $a^0 \partial_1 a^1 \partial_2 \partial_1 a^2 \partial_2 a^3$ is $\frac{4}{3}$. One obtains as the integrand in normal coordinates the expression

$$\frac{4}{3} \sum_{kl} a^0 \partial_\ell a^1 (\partial_\ell a^2 \partial_k^2 a^3 + \partial_k \partial_\ell a^2 \partial_k a^3 - \partial_k a^2 \partial_k \partial_\ell a^3 - \partial_k^2 a^2 \partial_\ell a^3) \mathrm{d}v$$

which is invariantly written as

$$\frac{4}{3} a^0 \mathrm{d}a^1 \wedge \mathrm{d}(\star(\mathrm{d}a^2 \wedge \mathrm{d}a^3))$$

and gives (3.24). □

We let ρ be given by the following formula (which makes sense in general)

$$\rho(a^0, a^1, a^2, a^3, a^4) = \fint a^0 [D^2, a^1][D^2, a^2][D, a^3][D, a^4]D^{-6} - \fint a^0 [D, a^1][D^2, a^2][D^2, a^3][D, a^4]D^{-6}$$

$$+ \fint a^0 [D, a^1][D, a^2][D^2, a^3][D^2, a^4]D^{-6}$$

$$- \fint a^0 [D^2, a^1][D, a^2][D, a^3][D^2, a^4]D^{-6}. \tag{3.25}$$

By construction ρ is a Hochschild cocycle and $B_0 \rho$ is already cyclic. We shall now show that assuming a condition of the form (3.23) one can use a multiple of ρ to eliminate ψ and replace τ_0 by a *positive* Hochschild cocycle.

Positivity in Hochschild cohomology was defined in [4] as the condition

$$\int_\tau \omega \omega^* \geq 0, \quad \forall \omega \in \Omega^2, \tag{3.26}$$

where the adjoint ω^* is defined by

$$(a_0 \mathrm{d}a_1 \mathrm{d}a_2)^* = \mathrm{d}a_2^* \mathrm{d}a_1^* a_0^*, \quad \forall a_j \in \mathcal{A}.$$

It then follows easily (*cf.* [3] Chapter VI) that the Yang–Mills action functional fulfills

$$YM_\tau(A) \geq 0, \quad \forall A \in \Omega^1.$$

We let π be the representation of Ω^* given by

$$\pi(a^0 \mathrm{d}a^1 \ldots \mathrm{d}a^n) = a^0 [D, a^1] \ldots [D, a^n]$$

Lemma 3.11. *The following equality defines a positive Hochschild cocycle:*

$$\tau_+(a^0, a^1, a^2, a^3, a^4) = - \fint a^0 ([D^2, a^1][D, a^2] - [D, a^1][D^2, a^2])$$

$$\times ([D^2, a^3][D, a^4] - [D, a^3][D^2, a^4])D^{-6}.$$

Let F be the sign of D, $\omega_j \in \Omega^2$ then

$$\int_{\tau_+} \omega_1 \omega_2 = \fint (F\pi(\omega_1)F - \pi(\omega_1))(F\pi(\omega_2)F - \pi(\omega_2))D^{-4}. \tag{3.27}$$

Proof. By construction τ_+ is a Hochschild cocycle. The right hand side of (3.27) fulfills (3.26) since for two forms one has $\pi(\omega^*) = (\pi(\omega))^*$. Thus it is enough to prove (3.27). By bilinearity we can assume that $\omega_1 = a^0 \mathrm{d}a^1 \mathrm{d}a^2$ and

$\omega_2 = \mathrm{d}a^3\mathrm{d}a^4b$. Moreover since $[F, b]$ is of the same order as D^{-1} we can absorb b in a^0 *i.e.* assume $b = 1$. Let us show that modulo bounded operators

$$[D, \pi(\omega_1)] = a^0([D^2, a^1][D, a^2] - [D, a^1][D^2, a^2]).$$

Indeed the left hand side is

$$[D, a^0][D, a^1][D, a^2] + a^0(D[D, a^1] + [D, a^1]D)[D, a^2] - a^0[D, a^1](D[D, a^2] + [D, a^2]D)$$
$$= [D, a^0][D, a^1][D, a^2] + a^0([D^2, a^1][D, a^2] - [D, a^1][D^2, a^2]).$$

Moreover, modulo bounded operators has

$$[D, \pi(\omega_1)] = [F, \pi(\omega_1)]|D| + F[|D|, \pi(\omega_1)] \sim [F, \pi(\omega_1)]|D|.$$

This gives

$$\int_{\tau_+} \omega_1\omega_2 = - \fint [F, \pi(\omega_1)]|D|[F, \pi(\omega_2)]|D|D^{-6} = - \fint [F, \pi(\omega_1)][F, \pi(\omega_2)]D^{-4}$$

and using $F^2 = 1$ yields (3.27). $\quad\square$

Proposition 3.12. *Let us assume that with φ given by (3.8) one has, for some $\mu \in \mathbb{R}$,*

$$\varphi = \mu \fint ([D^2, a^0][D, a^1] - [D, a^0][D^2, a^1])([D^2, a^2][D, a^3] - [D, a^2][D^2, a^3])D^{-6}. \tag{3.28}$$

(1) *With the notations of [Proposition 3.7](#) one has*
  - $\psi - \frac{\mu}{2}B_0\rho = 0$
  - $\tau = \tau_0 - \mu\rho$ *is equal to* $2\mu\tau_+$
(2) *Eq. (3.19) holds after the replacement $\tau_0 \to \tau$, $\psi \to 0$ and τ is positive if $\mu > 0$.*

Proof. Let φ be given by (3.8) and

$$\varphi^\lambda(a^0, a^1, a^2, a^3) = \varphi(a^1, a^2, a^3, a^0).$$

One has

$$\varphi + \varphi^\lambda = -B_0\tau_0. \tag{3.29}$$

Indeed by (3.11) one has

$$\fint [D, a^0a^1]D^{-1}[D, a^2]D^{-1}[D, a^3]D^{-1} = 0$$

which gives (3.29). We thus get $\psi = \varphi + \frac{1}{2}B_0\tau_0 = \frac{1}{2}(\varphi - \varphi^\lambda)$ and we can compute the right hand side using (3.28). One then gets using (3.25), that $\varphi - \varphi^\lambda = \mu B_0\rho$ which proves the first assertion.

To prove the second we now compute $\tau_0 - \mu\rho$. By [Lemma 3.1](#)(2) and (3.29) one gets

$$\tau_0 = -\frac{1}{2}b(\varphi + \varphi^\lambda)$$

and one can use (3.28) to compute the right hand side. The computation gives

$$(\tau_0 - \mu\rho)(a^0, a^1, a^2, a^3, a^4)$$
$$= -2\mu \fint a^0([D^2, a^1][D, a^2] - [D, a^1][D^2, a^2])([D^2, a^3][D, a^4] - [D, a^3][D^2, a^4])D^{-6}. \quad\square \tag{3.30}$$

In the Riemannian case one has $\mu = \frac{1}{6}$ by (3.23), and thus one gets

$$\tau(a^0, a^1, a^2, a^3, a^4) = -\frac{8}{3}\lambda \int_M \langle a^0\mathrm{d}a^1 \wedge \mathrm{d}a^2, \mathrm{d}a^3 \wedge \mathrm{d}a^4 \rangle$$

so that since $\lambda = \frac{1}{8\pi^2}$ one gets

$$\frac{1}{4}\int_\tau (\mathrm{d}A + A^2)^2 = \frac{1}{24\pi^2}\int_M \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\sqrt{g}\mathrm{d}^4x$$

which agrees with the direct computation using the Seeley–De Witt coefficients [1,7,8].

4. Open questions

We shall briefly formulate two important questions which are left open in the generality of the above framework.

4.1. Triviality of ψ

It is true under mild hypothesis that the vanishing of the pairing with the K_1-group

$$\langle\psi, u\rangle = 0, \quad \forall u \in K_1(\mathcal{A}),$$

implies that the cyclic cocycle ψ is homologous to zero,

$$\psi \in BZ^4(\mathcal{A}, \mathcal{A}^*).$$

Thus one can in any such case eliminate the Chern–Simons term using Proposition 3.7(2). We have not been able to find an example where ψ does not belong to the image of B and it could thus be that $\psi \in BZ^4(\mathcal{A}, \mathcal{A}^*)$ holds in full generality.

4.2. Positivity

In a similar manner the freedom given by Proposition 3.7 should be used to replace the Hochschild cocycle τ_0 by a *positive* Hochschild cocycle τ.

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